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Early Basic Foundations of Modern Integral Calculus

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ABSTRACT
The present article aims to bring the early basic foundations of modern integral calculus in front of the mathematics teachers and students especially in higher education. This contains the definitions of elementary - nonelementary functions; range and difficulty of problem of indefinite integration; existence of integrals; and lack of notations of nonelementary functions etc. In addition to this Bernoulli's conjecture; Laplace's theorem; Liouville's theorem; Chebyshev's theorem; Hardy's theorem; Inverse function theorem, etc. have also been presented with their applications.

Key Words: Elementary Function, Non-elementary Function, Indefinite Integration.

INTRODUCTION

The knowledge of teachers plays an important role in teaching and the basic theory of the subject makes the pillar in understanding it. To make the calculus students friendly, many attempts have been made to improve the understanding of the subject and the research scope while teaching it. Rasmussen et al. (2014) have identified four trends in the calculus research literature and discussed the gaps in literature and the new areas of calculus research needed. Swiden et al. (2014) have identified the objectification processes involved in making sense of the concept of an indefinite integral. Kouropatov et al. (2014) proposed an approach to the integral concept for advanced high school students and provide evidence for the potential of this approach to support students in acquiring an in depth view of the integral. Torner et al. (2014) noticed a reduction in the content of calculus and a more informal approach. Eichler et al. (2014) focused the belief systems towards teaching calculus of mathematics teachers. But authors both writers and researchers did not discuss the basic foundations of the integral calculus, which makes the subject complete. There are many unsolved problems in integral calculus, which have not been discussed by the authors and even by mathematics teachers in teaching it. These also do not appear in the calculus or integral calculus textbooks.

For example, why does the indefinite integral of $\exp(x^2)$ not exist? We know that the indefinite integral of cosx is sinx + c, is based on the fact that the differential coefficient of sinx+c is cosx and integration is the reverse process of differentiation. But if we ask what is the indefinite integral of (sinx)/x? We shall not find any reason except that there does not exist any function, which when differentiated gives (sinx)/x. Therefore we cannot integrate (sinx)/x in indefinite integral sense. This is only due to the lack of basic concepts and foundations of integral calculus. Recently Yadav (2012) have used the basic concepts and foundations of integral calculus to study indefinite nonintegrable functions.

Basic Concepts of Integral Calculus

The study of integral calculus is incomplete without the following concepts:

Elementary Function: A function is called an elementary function if it can be written as y=f(x), where f(x) represents an expression formed by combining a finite collection of powers of x, trigonometric functions, hyperbolic functions, exponentials, logarithms, inverse trigonometric functions, and inverse hyperbolic functions together with additions, subtractions, multiplications, divisions, powers, and compositions.

Example: $f(x)=\sin x+x$, $f(x)=5x^2+2x-e^x$, etc.

A function which is not elementary is known as a non-elementary function. Not all functions are elementary. The most common example of a non-elementary function is a piecewise-defined function. Well known examples of non-elementary functions are:

$$f(x)=\begin{cases}0\ if\ x\ is\ a\ rational\ number\\ 1\ if\ x\ is\ an\ irrational\ number\end{cases},\ f(x)=[x],\ \text{etc.}$$

Other types of non-elementary functions arise in calculus as "anti-derivatives" or 'indefinite integral' of an elementary function. Because the differentiation of an elementary function is always an elementary function, where as its indefinite integral may or may not be elementary. The non-elementary functions arising due to the integration have been named as 'indefinite nonintegrable functions' by Yadav to make it clear that it was due to indefinite integration.

Example: The indefinite integrals $\int \frac{e^x}{x}dx$, $\int \sin(x^2)dx$ are not elementary, i.e. non-elementary.

Range and Difficulty of Problem of Indefinite Integration

The indefinite integration of an elementary function f(x) is defined as a solution F(x), composed of elementary functions, such that $F'(x)=f(x)$. In mathematical symbol, it is denoted by

$$F(x)=\int_a^x f(t)dt=\int f(x)dx+k$$

Which on differentiation gives $F'(x)=f(x)$, where the constant of integration k corresponds to the value of the integral for the lower limit a suppose that f(x) belongs to a special class of functions S. We may ask whether F(x) is itself a member of S, or can be expressed, according to some simple standard mode of expression, in terms of functions which are members of S.

The range and difficulty of the problem of indefinite integration will depend upon the choice of:

(i) a class of functions, and

(ii) a standard mode of expression.

We shall take S to be the class of elementary functions, and our mode of expression to be that of explicit expression in finite terms.

Non-elementary or Indefinite Nonintegrable Functions in Integral Calculus

An indefinite integral of an elementary function is either an elementary function or can be expressed in terms of elementary functions in finite number of steps. If we say that an indefinite integral $\int f(x)dx$ is elementary (or integrable) it means that its integral exists and can be expressed in terms of elementary functions in closed form.

Those functions whose indefinite integrals are neither elementary nor can be expressed in terms of elementary functions are classically known as non-elementary. To make it clear that they arise due to integration, we call them nonintegrable functions or indefinite nonintegrable functions.

Example: $\int e^x dx = e^x + k$ and $\int \frac{dx}{1+x^2} = \tan^{-1} x + k$ are elementary; whereas $\int \frac{\sin x}{x} dx$ and $\int \sin(x^2) dx$ are non-elementary.

Existence of Integrals and Lack of Notations of Functions

The integral of an elementary function is the assertion of the Fundamental Theorem of Calculus: Every continuous function has an anti-derivative. Although it is not closely bound up with the assumption that the integrand is continuous, it may be extended to wide classes of functions with discontinuities.

But there is no guarantee that we can find a formula for an anti-derivative in terms of elementary functions like sine, cosine, logarithm, and so forth. There are elementary functions which have anti-derivatives but they cannot be expressed in terms of elementary functions due to the lack of notations of those functions.

Example: The error function $\int e^{-x^2} dx$, the exponential integral $\int \frac{e^x}{x} dx$, the sine integral $\int \frac{\sin x}{x} dx$, the cosine integral $\int \frac{\cos x}{x} dx$, etc.

Basic Theorems of Integral Calculus

The search for algorithm for elementary and non-elementary integral of elementary functions has been the subject of many efforts in the past. Some of them are as follows:

John Bernoulli's Conjecture (1712): The integral of any rational function is expressible in term of other rational functions, trigonometric functions, and logarithmic functions.

Example: $\int \frac{dx}{1+x^2} = \tan^{-1} x + k$, $\int \frac{2x dx}{1+x^2} = \ln(1+x^2) + k$, $\int \frac{-dx}{x^2} = \frac{1}{x} + k$.

Laplace's Theorem (1812): "A rational function has an anti-derivative and its integral is always an elementary function. In general it is composed of two parts: one of a rational function and the other the transcendental part or logarithmic part".

Example: $\int \frac{x dx}{1+x} = x - \ln(1+x) + k$, $\int \frac{(x^2+1)^2 + x}{x(x^2+1)} dx = \frac{x^2}{2} + \ln|x| + \frac{i}{2} \ln\left|\frac{x+i}{x-i}\right| + k$, etc.

Liouville's Theorems:

In 1833 **Joseph Liouville** based his work on the fact that the derivative of an elementary function is again an elementary function created a framework for constructive integration by finding out when indefinite integrals of elementary functions are again elementary functions. He introduced a theorem, which is reminiscent of Laplace's theorem, now known as:

Liouville's First Theorem on Integration:

If an algebraic function is integrable in finite terms, its anti-derivative is the finite sum of an algebraic function and the logarithms of algebraic functions. In mathematical symbols, if f(x) is an algebraic function of x and if $\int f(x) dx$ is elementary, then

$$\int f(x)dx = U_o + \sum_{i=1}^{n} C_i \log(U_i)$$

Where the C_i's are constants and the U_i's are algebraic functions of x.

In 1835, he generalized this theorem to several variables and gave:

Strong Liouville theorem:

(a) If F is an algebraic function of $x, y_1, y_2, ..., y_m$, where $y_1, y_2, ..., y_m$, are functions of x whose derivatives $dy_1/dx, dy_2/dx, ..., dy_m/dx$ are rational functions of $x, y_1, y_2, ..., y_m$, then

$$\int F(x, y_1, y_2, ..., y_m)dx$$

is elementary if and only if

$$\int F(x, y_1, y_2, ..., y_m)dx = U_0 + \sum_{j=1}^{n} C_j \log(U_j)$$

Where the C_j's are constants and the U_j's are algebraic functions of $x, y_1, y_2, ..., y_m$.

(b) If $F(x, y_1, y_2, ..., y_m)$ is a rational function and $dy_1/dx, dy_2/dx, ..., dy_m/dx$ are rational functions of $x, y_1, y_2, ..., y_m$, then the U_j's in part (a) must be rational functions of $x, y_1, y_2, ..., y_m$.

In the same year 1835, he found the special case of this theorem, which gives the necessary and sufficient conditions for the existence of elementary function of some special functions.

Strong Liouville theorem (special case):

If f(x) and g(x) are rational functions with g(x) non-constant, then

$$\int f(x)e^{g(x)}dx$$

is elementary if and only if there exists a rational function R(x) such that

$$f(x) = R'(x) + R(x)g'(x).$$

For any such R(x), $R(x)e^{g(x)}$ is an elementary anti-derivative of f(x).

By applying the above theorems he proved that the following integrals

$$\int \frac{dx}{\sqrt{P(x)}},\ \int e^{x^2}dx,\ \int \frac{e^x}{x}dx,\ \int e^{-x^2}dx,\ \int \frac{e^{-x}}{x}dx,\ \int \frac{\sin x}{x}dx,\ \int \frac{\cos x}{x}dx,\ \int \frac{dx}{\log x}$$

Cannot be expressed in terms of elementary functions.

There are two important properties obtained from the special case of strong Liouville theorem:

Property I: $\int x^{2n}e^{ax^2}dx$ for n an integer, is non-elementary for $a \neq 0$.

For n=0 and a=-1, this is the error function. By this property, he proved the following non-elementary functions:

$$\int \sqrt{\log x}\,dx = \int 2t^2e^{t^2}dt,\quad \int \frac{1}{\sqrt{\log x}}dx = \int 2e^{t^2}dt,\quad \int \frac{e^{ax}}{\sqrt{x}}dx = \int 2e^{at^2}dt$$

Property II: $\int x^{-n} e^{cx} dx$ for n a positive integer and c a nonzero constant, is non-elementary.

By this property he proved the following non-elementary functions:

$$\int e^{e^x} dx = \int \frac{e^t}{t} dt, \quad \int \frac{1}{\log x} dx = \int \frac{e^t}{t} dt,$$

$$\int \log(\log x) dx = x \log(\log x) - \int \frac{1}{\log x} dx, \quad \int \frac{\sin x}{x} dx = \operatorname{Im}\left(\int \frac{e^{ix}}{x} dx \right).$$

Chebyshev's Theorem:

In 1853 **P. L. Chebyshev** worked in the area of integration of specific forms of algebraic functions closely associated with the work of Abel and Liouville, and presented the theorem:

P. L. Chebyshev's Theorem: If p, q, and r are rational numbers and a, b are real numbers with $a, b, r \neq 0$, then

$$\int x^p (a + bx^r)^q dx$$

is elementary if and only if at least one of

$$\left(\frac{p+1}{r} \right), q, or \left(\frac{p+1}{r+q} \right) \text{is an integer.}$$

Applying this theorem, he showed that the following integrals

$$\int \left(1 + x^2\right)^{3/2} dx, \int \sqrt{1 + x^3} dx, \int \sqrt{1 + x^{-4}} dx, \int \sqrt{\sin x} dx \text{ and } \int \sqrt{\cos x} dx$$

are non-elementary functions.

He found the following corollaries from this theorem:

Corollary 1: If m and n are integers, then $\int (1 - x^n)^{1/m} dx$ is elementary if and only if $m = \pm 1$, or $n = \pm 1$, or $m = n = 2$, or $m = -n$.

Example: $\int \sin^m x \cos^n x dx$ and $\int \sqrt{\tan x} dx$ are elementary.

Corollary 2: He considered the integral

$$u = \int x^p (1 - x)^q dx,$$

where each of p and q is rational and not zero. He proved that, for the integral to be elementary, it is necessary and sufficient that at least one of p, q and p + q be an integer. It is known as Chebyshev's Integral.

Hardy's Theorem:

In 1905 **G. H. Hardy** found another special case of the strong Liouville theorem known as:

Liouville-Hardy Theorem: If f(x) is a rational function, then $\int f(x) \log x dx$ is elementary if and only if there exists a rational function g(x) and a constant C such that $f(x) = \frac{C}{x} + g'(x)$.

By this theorem, he showed that the integrals $\int \frac{\log x}{(x-a)} dx, a \neq 0$, $\int \frac{\log x}{(x^2+1)} dx$, and $\int \left(\sec^{-1} x\right)^2 dx$ are non-elementary.

At the last of this section, let us discuss the inverse function theorem, which was known to Liouville (1841) in writing his paper on Riccati equation. It also appears in the works of F. D. Parker (1955), J. H. Staib (1966), and in a recent note by E. Key (1994). It states that:

Inverse Functions Theorem: Let f(x) and $f^{-1}(x)$ be inverses of one another on some closed interval [a, b]. If f(x) and $f^{-1}(x)$ are elementary functions over [a, b], then $\int f(x)dx$ is elementary if and only if $\int f^{-1}(x)dx$ is elementary.

Example: $\int \sqrt{\log x}dx$ is non-elementary since the integral of the inverse function of its integrand, $\int e^{x^2}dx$, is non-elementary and $\int \frac{1}{\log x}dx$ is non-elementary since $\int e^{1/x}dx$ is non-elementary.

The above theorems are the basic properties of integral calculus, out of which the strong Liouville theorem and its special cases are the most important and useful.

Application of the Basic Theorems

Example: Integrate $1/(1+x^2)$ w. r. to x.

We have

$$\int \frac{dx}{1+x^2} = \int F[x,(1+x^2)]dx = \int F[x,y_1]dx, \text{ where } \left[\frac{dy_1}{dx} = 2x \in F\right]$$

By strong Liouville theorem

$$\int \frac{dx}{1+x^2} = U_0 + \sum_{j=1}^{n} C_j \log U_j \quad \text{……….. (1)}$$

On differentiating it we get, $\frac{1}{1+x^2} = U'_0 + \sum_{j=1}^{n} C_j \frac{U'_j}{U_j}$

Now, $U'_0 + \sum_{j=1}^{n} C_j \frac{U'_j}{U_j} = \frac{1}{(1+ix)(1-ix)} = \frac{1}{2}\left[\frac{1}{(1+ix)} + \frac{1}{(1-ix)}\right]$

$$= \frac{1}{2i}\left[\frac{(1+ix)'}{(1+ix)} - \frac{(1-ix)'}{(1-ix)}\right] = \frac{1}{2i}\left[\{\log(1+ix)\}' - \{\log(1-ix)\}'\right]$$

Where the sign (‘) denotes differentiation with respect to x. Comparing it with (1) we get $U_0' = 0$ therefore $U_0=K$ and $C_1=(1/2i)$, $C_2=(-1/2i)$, $U_1=(1+ix)$, $U_2=(1-ix)$.

Therefore, $\int \frac{dx}{1+x^2} = \frac{1}{2i}\left[\{\log(1+ix)\} - \{\log(1-ix)\}\right] + K$

$$= \frac{1}{2i}\log\frac{(1+ix)}{(1-ix)} + K = -\frac{1}{2i}\log\frac{(1-ix)}{(1+ix)} + K = \frac{i}{2}\log\frac{(i+x)}{(i-x)} + K = \tan^{-1}x + K.$$

Example: Find $\int xe^x dx$.

By strong Liouville theorem

$$x = R'(x) + R(x) \quad \text{……………. (1)}$$

Let R(x)=p/q; gcd(p, q)=1. Therefore from (1)

$$x = \frac{qp' - pq'}{q^2} + \frac{p}{q}$$

$$\Rightarrow xq^2 = qp' - pq' + pq \quad \Rightarrow q[p' - xq + p] = pq'$$

$$\Rightarrow p' - xq + p = \frac{pq'}{q} \; \ldots\ldots\ldots (2)$$

which means either q divides p, or q divides q'. But q cannot divide p, which implies that q divides q' and therefore q is a constant. Without loss of generality we may let q=1 i. e. R(x)=p. Then from (2)

p'-x+p=0 i. e. p'+p=x ……. (3)

Since p is a polynomial, let p=ax+b, as degree of p(x) cannot be greater than 1. Then from (3) we have

a+ax+b=x i. e. ax+(a+b)=x which implies a=1 and a+b=0 which gives b=-1.

Therefore p(x)=x-1, i. e. R(x)=(x-1).

So we have

$$\int xe^x dx = R(x)e^x = (x-1)e^x = (x-1)e^x + K$$

Example: Find $\int 2xe^{x^2} dx$.

By strong Liouville theorem

2x=R'(x)+2xR(x), where R=p/q with gcd(p, q)=1. Therefore

2x=R'(x)+2xR(x) implies that $2x = \frac{qp' - pq'}{q^2} + 2x\frac{p}{q}$

$$\Rightarrow 2xq^2 = qp' - pq' + 2xpq \quad \Rightarrow q[p' - 2xq + 2xp] = pq'$$

$$\Rightarrow p' - 2xq + 2xp = \frac{pq'}{q} \; \ldots\ldots\ldots (1)$$

which implies either q/p or q/q'. But q cannot divide p because gcd(p, q)=1. Therefore q/q' which means that q is a constant. Without loss of generality we may assume that q=1. Therefore from (1) p'-2x+2xp=0 i.e. p'+2xp=2x. Comparing the degrees of x, we find that p=1 and p'=0, which implies that R(x)=1. Therefore

$$\int 2xe^{x^2} dx = R(x)e^{x^2} = 1.e^{x^2} = e^{x^2} + K$$

Example: Show that the integral $\int \frac{e^{ax^2}}{x} dx, a \neq 0$ is non-elementary.

Proof: We have

$$\int \frac{e^{ax^2}}{x} dx = \int \frac{2axe^{ax^2}}{2ax^2} dx$$

Putting ax^2=z we get

$$\int \frac{2axe^{ax^2}}{2ax^2} dx = \frac{1}{2}\int \frac{e^z}{z} dz = \frac{1}{2}\int z^{-1}e^z dz$$

Which is non-elementary from Property II. Hence the given integral is non-elementary.

Example: Show that the integral $\int \frac{\sin x}{x} dx$ is non-elementary.

Proof: We have using Euler's identity

$$\int \frac{\sin x}{x} dx = \text{img}\left[\int \frac{e^{ix}}{x} dx\right]$$

Taking g(x)=ix and f(x)=1/x, we see that it is elementary if and only if there exists a rational function R(x) which satisfies the identity

$$R'(x) + iR(x) = \frac{1}{x} \Rightarrow R'(x) = \frac{1}{x} \ \& \ R(x) = 0$$

Which is impossible. Hence the given function is non-elementary.

Similarly we can solve all most all the examples of indefinite integrals by the help of the theorems, corollaries, properties, and their special cases.

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